A Subatomic Proof System for Decision Trees

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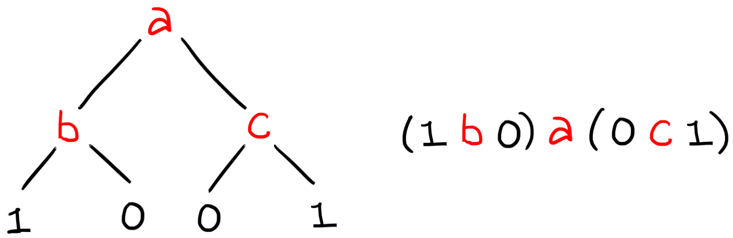
Overview

- Present a novel proof system for a conservative extension of classical propositional logic which includes decision trees
- Using the subatomic methodology[1] to efficiently design (even discover) new deep inference proof systems
- An extremely simple cut elimination procedure

[1] Developed by Aler Tubella and Guglielmi

Decision Trees

- A decision tree (DT) is a binary tree of conditionals representing a boolean function $f\colon \{0,1\}^{|\mathcal{A}|} \to \{0,1\}$, for a set of variables $\mathcal{A} = \{a, b, c, \ldots\}$
- Each node labelled by a boolean variable and leaves labelled by 0 or 1



- Evaluation:

$$[\![B \text{ a } C]\!]_X = \begin{cases} [\![B]\!]_X & \text{if } X(a) = 0 \\ [\![C]\!]_X & \text{if } X(a) = 1 \end{cases}, \quad \text{for } X\colon \mathcal{A} \to \{0,1\}$$

Deep Inference

- Allows free composition of derivations, horizontally via any connective of a given language, as well as via inference rules

$$\begin{array}{c} A \\ \| \\ C \end{array} \quad ::= \quad A \;\mid\; \boxed{\begin{array}{c} A_1 \\ \| \\ C_1 \end{array}} \;\beta\; \boxed{\begin{array}{c} A_2 \\ \| \\ C_2 \end{array}} \;\mid\; \begin{array}{c} A \\ \| \\ r\,\dfrac{B}{B'} \\ \| \\ C \end{array}$$

for any connective β and inference rule r.

- Boxes are 2-dimensional brackets

Generalized Subatomic Language

- Suppose we wish to prove:

$$(0 \text{ a } (1 \text{ b } 0)) \to (1 \text{ b } (0 \text{ a } 1))$$

- To express implication, mix the language of DTs with propositional connectives
- Given a set of atoms $\mathcal{A}$, define our set of formulae:

$$\mathcal{F} \quad ::= \quad 1 \mid 0 \mid (\mathcal{F} \wedge \mathcal{F}) \mid (\mathcal{F} \vee \mathcal{F}) \mid (\mathcal{F} \ \mathcal{A} \ \mathcal{F})$$

- Atoms a $\in \mathcal{A}$ are treated as de Morgan self-dual variable left/right projections
- We can express classical propositional formulae using the embedding:

$$a \quad \rightsquigarrow \quad (0 \text{ a } 1) \qquad \bar{a} \quad \rightsquigarrow \quad (1 \text{ a } 0)$$

- Formulae in the image of this embedding (i.e. those without 'nesting' of atoms) are *interpretable* in classical propositional logic

Generalized Subatomic Language

- Suppose we wish to prove:

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- To express implication, mix the language of DTs with propositional connectives

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- Atoms $\mathsf{a} \in \mathcal{A}$ are treated as de Morgan self-dual variable left/right projections
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- Formulae in the image of this embedding (i.e. those without 'nesting' of atoms) are *interpretable* in classical propositional logic

- We can prove $(1 \text{ a } (0 \text{ b } 1)) \vee (1 \text{ b } (0 \text{ a } 1))$ by case analysis:

$$\bar{a} \longrightarrow (1 \text{ a } (0 \text{ b } 1)) \quad i.e. \quad (0 \text{ a } 1) \vee (1 \text{ a } (0 \text{ b } 1))$$

$$a \longrightarrow (1 \text{ b } (0 \text{ a } 1)) \quad i.e. \quad (1 \text{ a } 0) \vee (1 \text{ b } (0 \text{ a } 1))$$

$$\boxed{{\scriptstyle =}\dfrac{1}{{\scriptstyle \vee\text{ă}}\dfrac{\boxed{{\scriptstyle =}\dfrac{1}{0 \vee 1}} \text{ a } \boxed{{\scriptstyle =}\dfrac{1}{1 \vee 0}}}{(0 \text{ a } 1) \vee (1 \text{ a } 0)}}} \qquad\qquad \dfrac{1}{a \vee \bar{a}}$$

- We can still express identity and cut as instances of more general inference rules acting on certain interpretable formulae

$$\vee\text{ă}\,\dfrac{(A \vee B) \text{ a } (C \vee D)}{(A \text{ a } C) \vee (B \text{ a } D)}$$

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$$a \longrightarrow (1\ \mathsf{b}\ (0\ \mathsf{a}\ 1)) \quad i.e. \quad (1\ \mathsf{a}\ 0) \vee (1\ \mathsf{b}\ (0\ \mathsf{a}\ 1))$$

$$\vee\check{a}\ \dfrac{=\dfrac{1}{(0 \vee 1)\ \mathsf{a}\ (1 \vee 0)}}{(0\ \mathsf{a}\ 1) \vee \boxed{1\ \mathsf{a}\ \boxed{\begin{matrix}0\\ \omega_1 \,\| \\ 0\ \mathsf{b}\ 1\end{matrix}}}}$$

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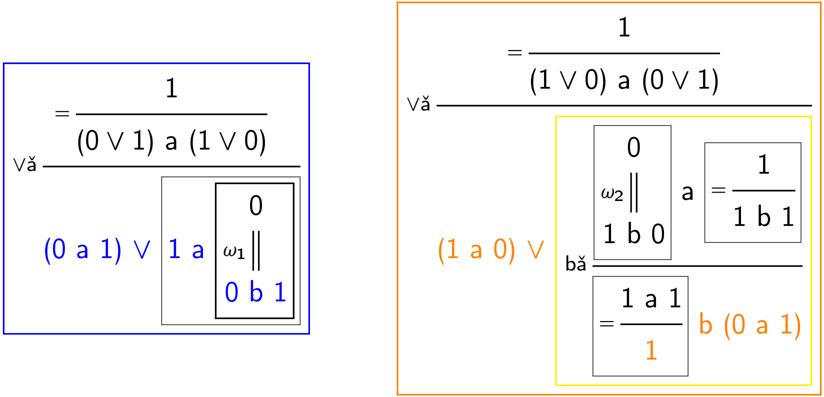

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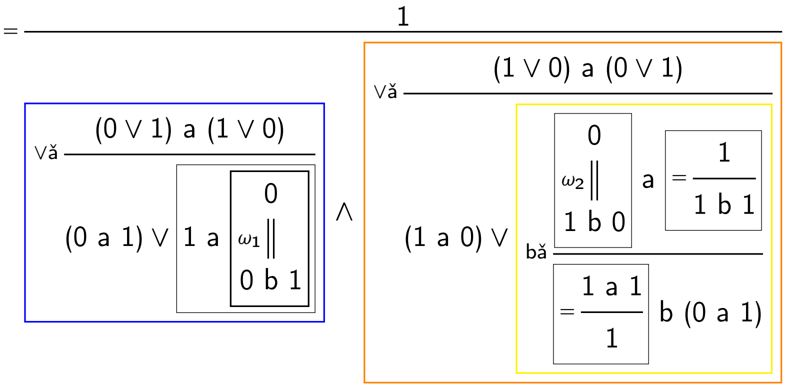

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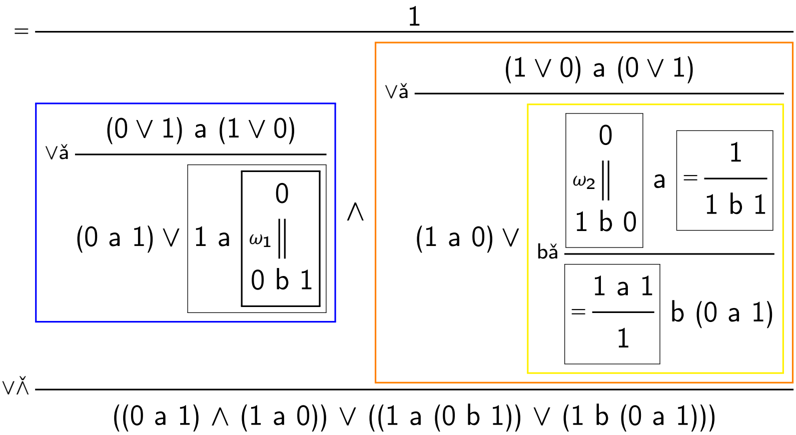

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$$
=\,\dfrac{\vee\check{\wedge}\,\dfrac{=\,\dfrac{1}{\boxed{\vee\check{a}\,\dfrac{(0 \vee 1)\ a\ (1 \vee 0)}{(0\ a\ 1) \vee \boxed{1\ a\ \boxed{\begin{array}{c}0\\ \omega_1\,\|\\ 0\ b\ 1\end{array}}}}} \wedge \boxed{\vee\check{a}\,\dfrac{(1 \vee 0)\ a\ (0 \vee 1)}{(1\ a\ 0) \vee \boxed{b\check{a}\,\dfrac{\boxed{\begin{array}{c}0\\ \omega_2\,\|\\ 1\ b\ 0\end{array}}\ a\ \boxed{=\,\dfrac{1}{1\ b\ 1}}}{\boxed{=\,\dfrac{1\ a\ 1}{1}}\ b\ (0\ a\ 1)}}}}}}{\boxed{\text{cut}\,\dfrac{(0\ a\ 1) \wedge (1\ a\ 0)}{(0 \wedge 1)\ a\ (1 \wedge 0)}} \vee ((1\ a\ (0\ b\ 1)) \vee (1\ b\ (0\ a\ 1)))}}{(1\ a\ (0\ b\ 1)) \vee (1\ b\ (0\ a\ 1))}
$$

Subatomic Methodology

- A single rule shape can generate *all* the standard inference rules for a variety of logics (MALL, BV, classical, ...) - including all just demonstrated
- How? Consider atoms as connectives: $a \rightsquigarrow (0 \text{ a } 1)$ and $\bar{a} \rightsquigarrow (1 \text{ a } 0)$
- Regularity of rules: study of normalization is at once simplified and generalized
- Aler Tubella proves simple sufficient conditions for subatomic proof systems to enjoy cut elimination

The Rule Shape

- Given connectives α and β, we have dual instances of the shape: the *up* and *down* rules

$$\alpha\hat{\beta}\,\frac{(A\ \beta\ B)\ \alpha\ (C\ \hat{\beta}\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \alpha\ D)} \qquad \beta\check{\alpha}\,\frac{(A\ \beta\ B)\ \alpha\ (C\ \beta\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \check{\alpha}\ D)}$$

- Decorations $\hat{\cdot}$ and $\check{\cdot}$ shift connectives to their strong and weak counterparts respectively

Strong:	$\wedge$	$\&$	$\otimes$	a
Weak:	$\vee$	$\oplus$	$⅋$	a

The Rule Shape

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- Example:

$$\hat{\wedge} = \wedge \quad \check{\wedge} = \vee$$

$$\hat{\vee} = \wedge \quad \check{\vee} = \vee$$

$$\hat{\mathsf{a}} = \check{\mathsf{a}} = \mathsf{a}$$

- The system SKS^{sa} for classical logic is generated by a subset of these rules, for $\alpha, \beta \in \{\wedge, \vee\} \cup \mathcal{A}$

System SKS^{sa}

- We can still express identity, cut and (co-)contraction as instances of the linear rule shape acting on interpretable formulae

$$\vee\hat{a}\ \frac{(0\ \mathsf{a}\ 1) \vee (0\ \mathsf{a}\ 1)}{\boxed{=\frac{0 \vee 0}{0}}\ \mathsf{a}\ \boxed{=\frac{1 \vee 1}{1}}} \qquad \frac{a \vee a}{a} \qquad\qquad \wedge\hat{a}\ \frac{(0\ \mathsf{a}\ 1) \wedge (1\ \mathsf{a}\ 0)}{\boxed{=\frac{0 \wedge 1}{0}}\ \mathsf{a}\ \boxed{=\frac{1 \wedge 0}{0}}} \qquad \frac{a \wedge \bar{a}}{0}$$

$$\vee\check{a}\ \frac{\boxed{=\frac{1}{0 \vee 1}}\ \mathsf{a}\ \boxed{=\frac{1}{1 \vee 0}}}{(0\ \mathsf{a}\ 1) \vee (1\ \mathsf{a}\ 0)} \qquad \frac{1}{a \vee \bar{a}} \qquad\qquad \wedge\check{a}\ \frac{\boxed{=\frac{0}{0 \wedge 0}}\ \mathsf{a}\ \boxed{=\frac{1}{1 \wedge 1}}}{(0\ \mathsf{a}\ 1) \wedge (0\ \mathsf{a}\ 1)} \qquad \frac{a}{a \wedge a}$$

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$$\vee\hat{a}\,\frac{(A \text{ a } B) \vee (C \text{ a } D)}{(A \vee C) \text{ a } (B \vee D)} \qquad \wedge\hat{a}\,\frac{(A \text{ a } B) \wedge (C \text{ a } D)}{(A \wedge C) \text{ a } (B \wedge D)}$$

$$\vee\check{a}\,\frac{(A \vee B) \text{ a } (C \vee D)}{(A \text{ a } C) \vee (B \text{ a } D)} \qquad \wedge\check{a}\,\frac{(A \wedge B) \text{ a } (C \wedge D)}{(A \text{ a } C) \wedge (B \text{ a } D)}$$

- In system SKS^{sa} , we consider only interpretable formulae in order to maintain a correspondence with the standard proofs of system SKS

System SKS^{sa}

Assoc/comm.

Switch and Medial

Identity and Cut

(Co−)contraction

$$\vee\check{\vee}\,\frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)}$$

$$\vee\check{\wedge}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}$$

$$\vee\check{\mathsf{a}}\,\frac{(A \vee B)\ \mathsf{a}\ (C \vee D)}{(A\ \mathsf{a}\ C) \vee (B\ \mathsf{a}\ D)}$$

$$\wedge\hat{\wedge}\,\frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge\hat{\vee}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

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$$\wedge\check{\mathsf{a}}\,\frac{(A \wedge B)\ \mathsf{a}\ (C \wedge D)}{(A\ \mathsf{a}\ C) \wedge (B\ \mathsf{a}\ D)}$$

$$\mathsf{a}\check{\vee}\,\frac{(A\ \mathsf{a}\ B) \vee (C\ \mathsf{a}\ D)}{(A \vee C)\ \mathsf{a}\ (B \vee D)}$$

System SKS^{sa} + Duplicates

Assoc/comm.

Switch and Medial

Identity and Cut

(Co−)contraction

$$\vee\check{\vee}\ \frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} \qquad \wedge\check{\vee}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{a}\check{\vee}\ \frac{(A \mathbin{\mathsf{a}} B) \vee (C \mathbin{\mathsf{a}} D)}{(A \vee C) \mathbin{\mathsf{a}} (B \vee D)}$$

$$\vee\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}$$

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System SKS^{sa} + Duplicates + Derivable Rules

Assoc/comm.

Switch and Medial

Identity and Cut

(Co−)contraction

Derivable

$$\vee\check{\vee}\,\frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} \qquad \wedge\check{\vee}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{a}\check{\vee}\,\frac{(A \mathbin{\mathsf{a}} B) \vee (C \mathbin{\mathsf{a}} D)}{(A \vee C) \mathbin{\mathsf{a}} (B \vee D)}$$

$$\vee\check{\wedge}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \wedge\check{\wedge}\,\frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \vee D)} \qquad \mathsf{a}\check{\wedge}\,\frac{(A \mathbin{\mathsf{a}} B) \wedge (C \mathbin{\mathsf{a}} D)}{(A \wedge C) \mathbin{\mathsf{a}} (B \vee D)}$$

$$\vee\check{\mathsf{a}}\,\frac{(A \vee B) \mathbin{\mathsf{a}} (C \vee D)}{(A \mathbin{\mathsf{a}} C) \vee (B \mathbin{\mathsf{a}} D)} \qquad \wedge\check{\mathsf{a}}\,\frac{(A \wedge B) \mathbin{\mathsf{a}} (C \wedge D)}{(A \mathbin{\mathsf{a}} C) \wedge (B \mathbin{\mathsf{a}} D)}$$

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$$\wedge\hat{\vee}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \vee\hat{\vee}\,\frac{(A \vee B) \vee (C \wedge D)}{(A \vee C) \vee (B \vee D)} \qquad \mathsf{a}\hat{\vee}\,\frac{(A \vee B) \mathbin{\mathsf{a}} (C \wedge D)}{(A \mathbin{\mathsf{a}} C) \vee (B \mathbin{\mathsf{a}} D)}$$

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System DT^{sa}: All Rules Generated By One Shape!

Assoc/comm.

Switch and Medial

Identity and Cut

(Co−)contraction

Derivable

Reordering DTs

$$\vee\check{\vee}\ \frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} \qquad \wedge\check{\vee}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{a}\check{\vee}\ \frac{(A \mathsf{~a~} B) \vee (C \mathsf{~a~} D)}{(A \vee C) \mathsf{~a~} (B \vee D)}$$

$$\vee\check{\wedge}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \wedge\check{\wedge}\ \frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \vee D)} \qquad \mathsf{a}\check{\wedge}\ \frac{(A \mathsf{~a~} B) \wedge (C \mathsf{~a~} D)}{(A \wedge C) \mathsf{~a~} (B \vee D)}$$

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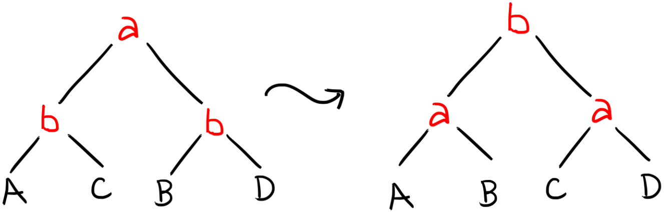
Reordering Decsion Trees



$$\text{a}\hat{\text{b}}\,\frac{(A \text{ b } B) \text{ a } (C \text{ b } D)}{(A \text{ a } C) \text{ b } (B \text{ a } D)}$$

- To include the new rule, we must consider a natural generalization of interpretable formulae

The Rule Shape

- Given connectives α and β, we define dual instances of the shape: the *up* and *down* rules

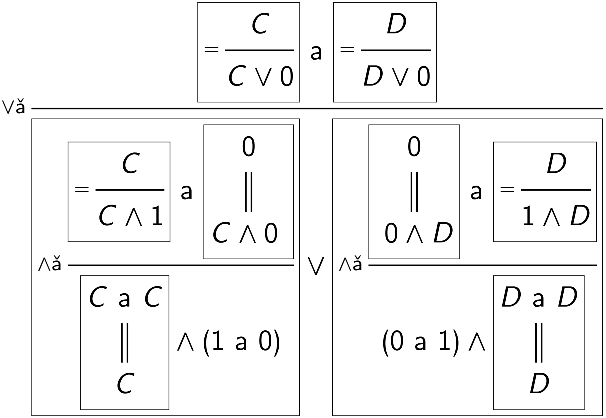
$$\alpha\hat{\beta}\,\frac{(A\ \beta\ B)\ \alpha\ (C\ \hat{\beta}\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \alpha\ D)} \qquad \beta\check{\alpha}\,\frac{(A\ \beta\ B)\ \alpha\ (C\ \beta\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \check{\alpha}\ D)}$$

- The system SKS^{sa} is generated by a subset of these rules, for $\alpha, \beta \in \{\wedge, \vee\} \cup \mathcal{A}$
- The system DT^{sa} is generated by all the rules, for $\alpha, \beta \in \{\wedge, \vee\} \cup \mathcal{A}$
- Thus, system DT^{sa} really is *just* the shape!

Completeness

- We can prove within the system the semantic equivalence:

$$C \text{ a } D \;\leftrightarrow\; (C \wedge (1 \text{ a } 0)) \vee ((0 \text{ a } 1) \wedge D)$$



- This construction is invertible, and its inversion is also cut-free

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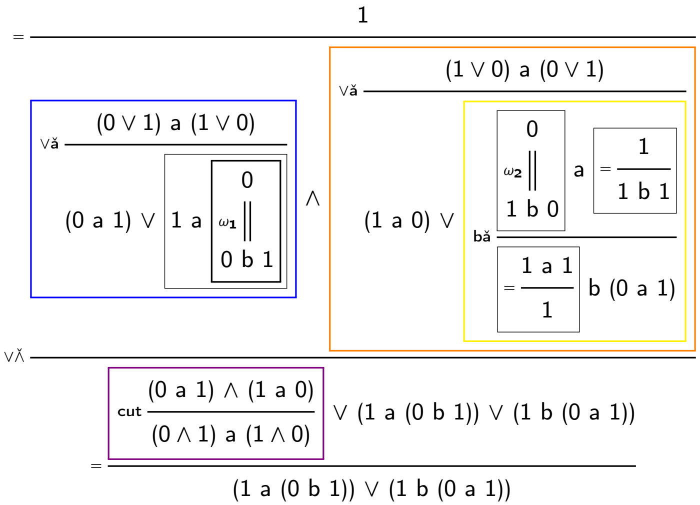
$$C \mathbin{\mathsf{a}} D \;\leftrightarrow\; (C \wedge (1 \mathbin{\mathsf{a}} 0)) \vee ((0 \mathbin{\mathsf{a}} 1) \wedge D)$$

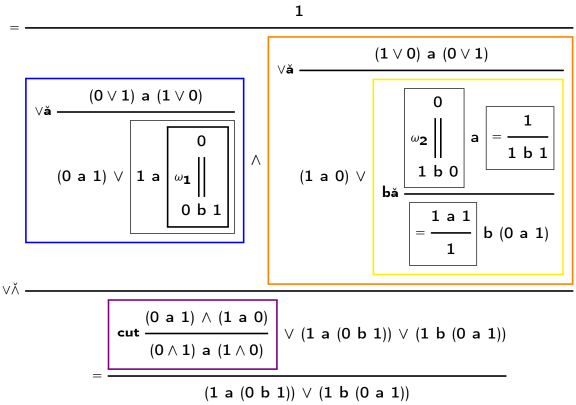
$$
{\scriptstyle \mathsf{a}\check{\vee}}\,\dfrac{
\left[{\scriptstyle \wedge\hat{\mathsf{a}}}\,\dfrac{\left[\begin{array}{c} C \\ \| \\ C \mathbin{\mathsf{a}} C \end{array}\right] \wedge (1 \mathbin{\mathsf{a}} 0)}{\left[= \dfrac{C \wedge 1}{C} \right] \mathbin{\mathsf{a}} \left[\begin{array}{c} C \wedge 0 \\ \| \\ 0 \end{array}\right]}\right]
\vee
\left[{\scriptstyle \wedge\hat{\mathsf{a}}}\,\dfrac{(0 \mathbin{\mathsf{a}} 1) \wedge \left[\begin{array}{c} D \\ \| \\ D \mathbin{\mathsf{a}} D \end{array}\right]}{\left[\begin{array}{c} 0 \wedge D \\ \| \\ 0 \end{array}\right] \mathbin{\mathsf{a}} \left[= \dfrac{1 \wedge D}{D} \right]}\right]
}{
\left[= \dfrac{C \vee 0}{C} \right] \mathbin{\mathsf{a}} \left[= \dfrac{0 \vee D}{D} \right]
}
$$

- This allows us to reduce completeness of DT^{sa} to that of SKS^{sa}, which is known.

Cut Elimination

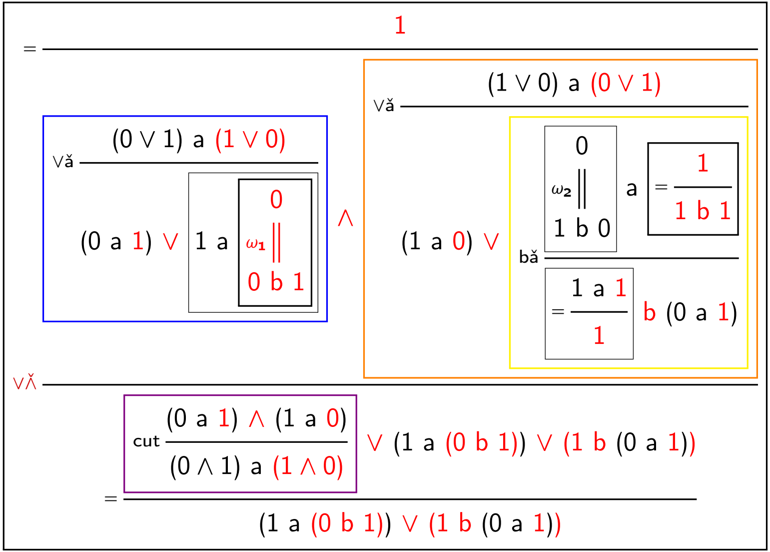


- We call a *cut on* a those inferences interpretable as a cut in SKS on atoms a and $\bar{a}$

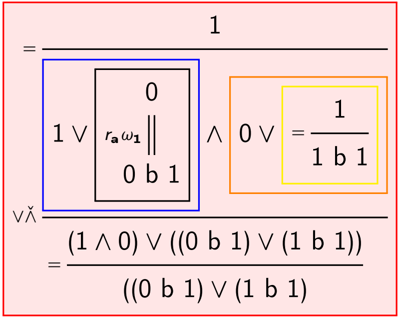


Definition (Informal)

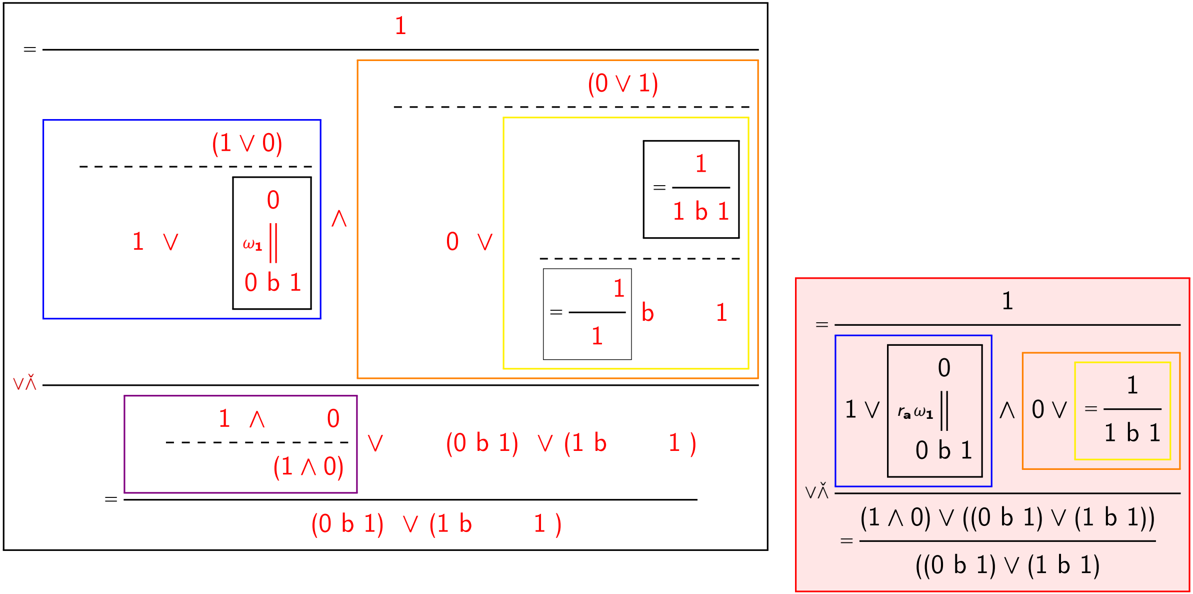
The *left* (*right*) *projection on* a of a derivation ϕ is a derivation $l_a \phi$ ($r_a \phi$) defined by replacing every occurence of B a C with B (C), i.e replace every atom a with the left (right) projection operator and simplify. Fix the broken inference rules in the obvious way.



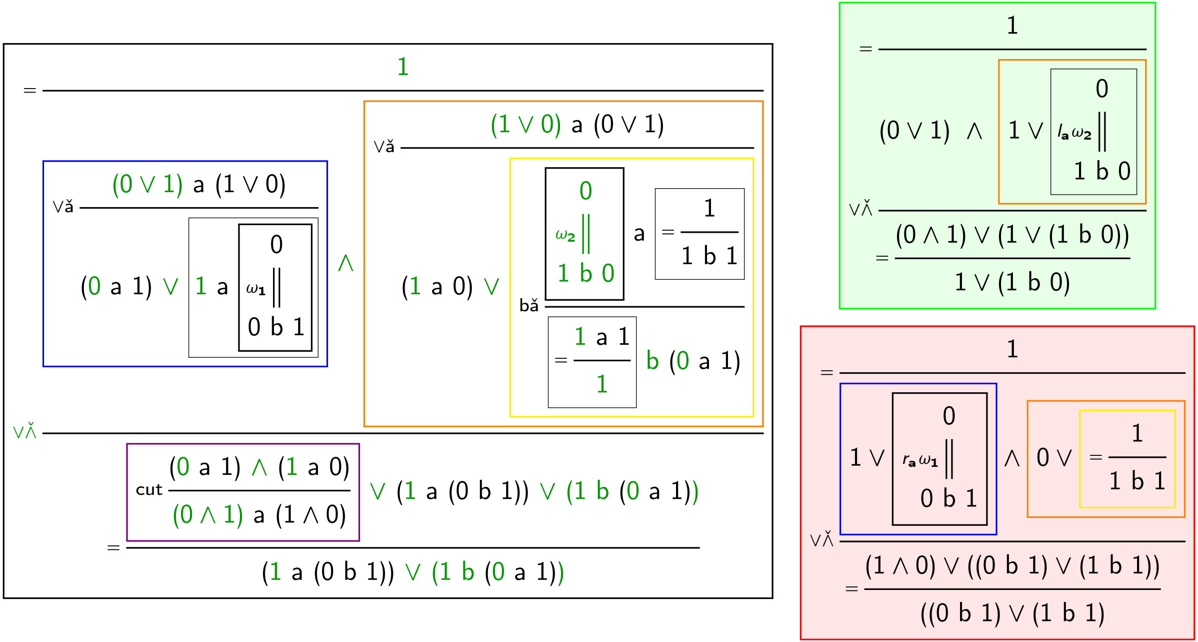
1
(1 ∨ 0) a (0 ∨ 1)
(0 ∨ 1) a (1 ∨ 0)
∨ă
(0 a 1) ∨ 1 a
0
ω₁
0 b 1
∧
∨ă
0
ω₂
1 b 0
a
1
1 b 1
bă
1 a 1
1
b (0 a 1)
(1 a 0) ∨
∨∧̌
(0 a 1) ∧ (1 a 0)
cut
(0 ∧ 1) a (1 ∧ 0)
∨ (1 a (0 b 1)) ∨ (1 b (0 a 1))
(1 a (0 b 1)) ∨ (1 b (0 a 1))



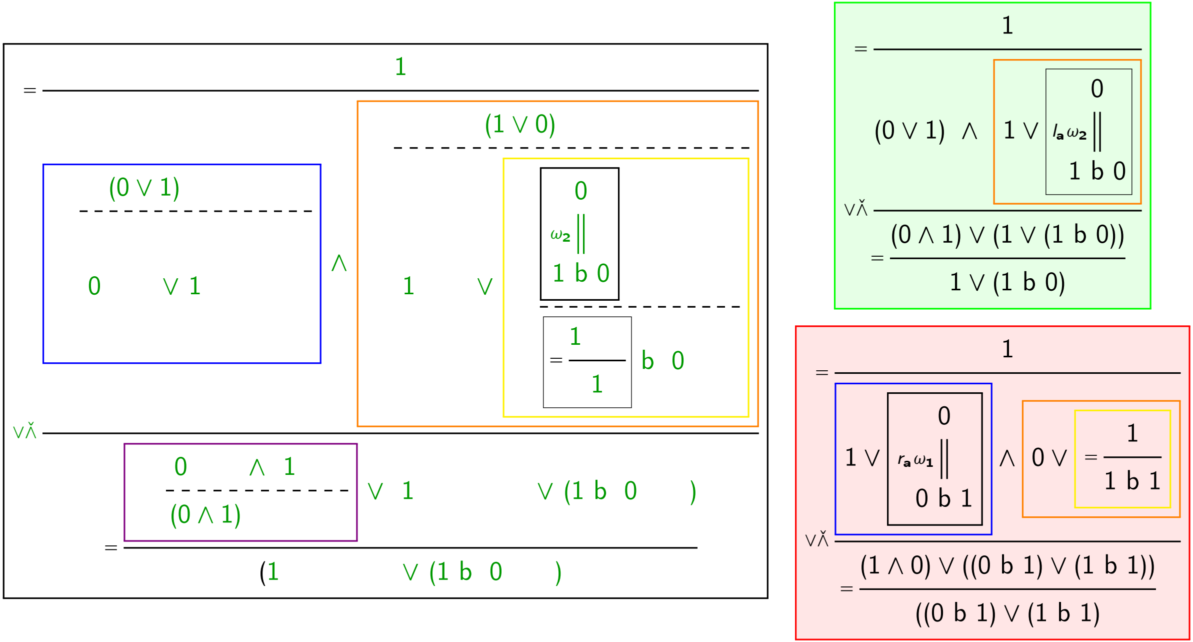
1
1 ∨
0
r_a ω₁
0 b 1
∧
0 ∨
1
1 b 1
∨∧̌
(1 ∧ 0) ∨ ((0 b 1) ∨ (1 b 1))
((0 b 1) ∨ (1 b 1))



1
(0 ∨ 1)
(1 ∨ 0)
0
1 ∨
ω₁
0 b 1
∧
0 ∨
1
1 b 1
1
1
b 1
1 ∧ 0
(1 ∧ 0)
∨ (0 b 1) ∨ (1 b 1)
(0 b 1) ∨ (1 b 1)
1
1 ∨
0
r_a ω₁
0 b 1
∧ 0 ∨
1
1 b 1
(1 ∧ 0) ∨ ((0 b 1) ∨ (1 b 1))
((0 b 1) ∨ (1 b 1))



1
(0 ∨ 1) a (1 ∨ 0)
(0 a 1) ∨ 1 a
0
ω₁
0 b 1
(1 ∨ 0) a (0 ∨ 1)
(1 a 0) ∨
0
ω₂
1 b 0
a
1
1 b 1
1 a 1
1
b (0 a 1)
cut
(0 a 1) ∧ (1 a 0)
(0 ∧ 1) a (1 ∧ 0)
∨ (1 a (0 b 1)) ∨ (1 b (0 a 1))
(1 a (0 b 1)) ∨ (1 b (0 a 1))
1
(0 ∨ 1) ∧ 1 ∨
0
l_a ω₂
1 b 0
(0 ∧ 1) ∨ (1 ∨ (1 b 0))
1 ∨ (1 b 0)
1
1 ∨
0
r_a ω₁
0 b 1
∧ 0 ∨
1
1 b 1
(1 ∧ 0) ∨ ((0 b 1) ∨ (1 b 1))
((0 b 1) ∨ (1 b 1)



1
(1 ∨ 0)
(0 ∨ 1)
0 ∨ 1
∧
1 ∨
0
ω2
1 b 0
1
1
b 0
∨Ǎ
0 ∧ 1
(0 ∧ 1)
∨ 1 ∨ (1 b 0)
(1 ∨ (1 b 0)
1
(0 ∨ 1) ∧ 1 ∨
0
la ω2
1 b 0
∨Ǎ
(0 ∧ 1) ∨ (1 ∨ (1 b 0))
1 ∨ (1 b 0)
1
1 ∨
0
ra ω1
0 b 1
∧ 0 ∨
1
1 b 1
∨Ǎ
(1 ∧ 0) ∨ ((0 b 1) ∨ (1 b 1))
((0 b 1) ∨ (1 b 1)

Cut Elimination

Theorem

The cut rule is admissible.

For every formula B, there exists a cut-free derivation: $\begin{array}{c} l_{\mathsf{a}}B \ \mathsf{a} \ r_{\mathsf{a}}B \\ \chi \Big\Vert \\ B \end{array}$

Cut Elimination

Theorem

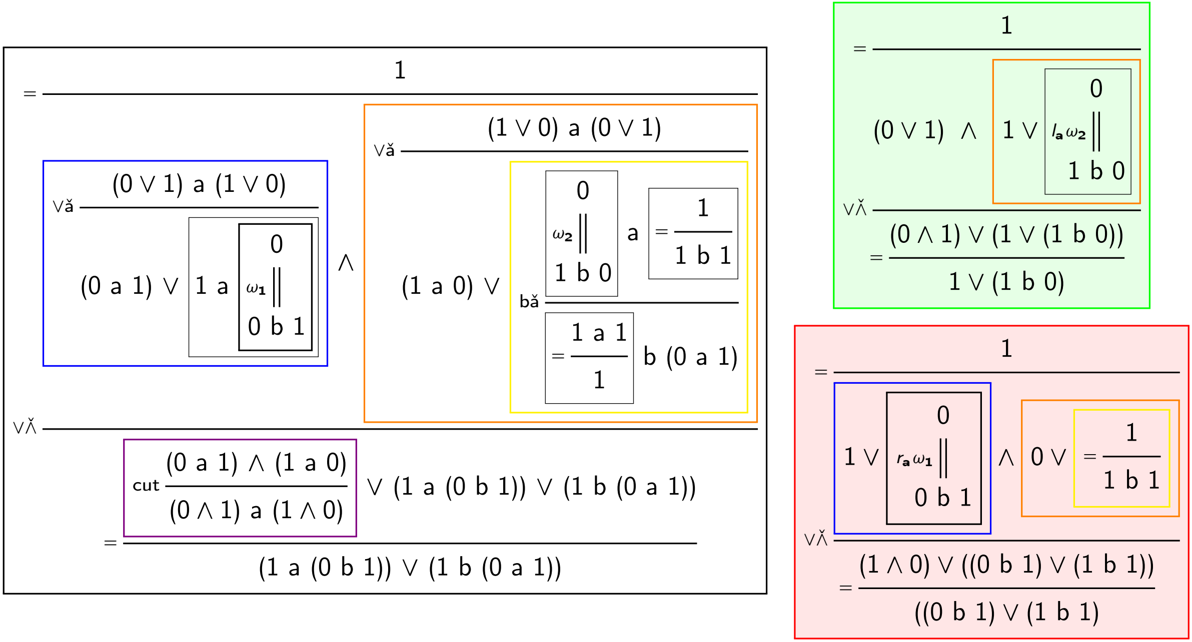
The cut rule is admissible.

For every formula B, there exists a cut-free derivation: $\begin{array}{c} l_a B \;\; a \;\; r_a B \\ \chi \Big\| \\ B \end{array}$

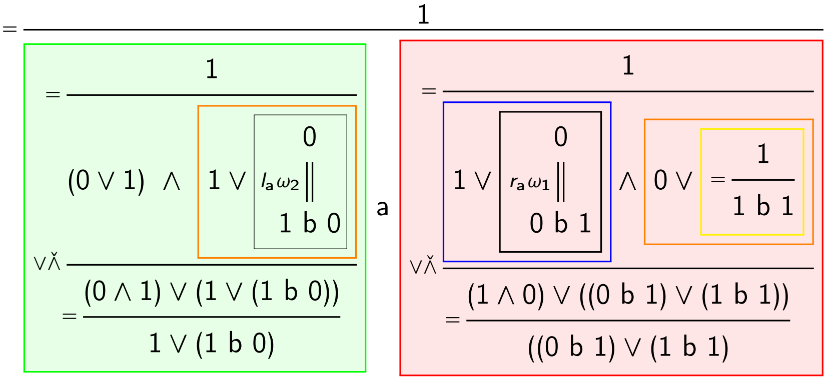
Given a proof ϕ of B, containing a cut on a, construct:

$$\begin{array}{c} 1 \\ \phi \Big\| \\ B \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \boxed{ = \dfrac{1}{\begin{array}{c} 1 \\ l_a \phi \Big\| \\ l_a B \end{array} \quad a \quad \begin{array}{c} 1 \\ r_a \phi \Big\| \\ r_a B \end{array}} } \\ \chi \Big\| \\ B \end{array}$$

Iterating this process yields a cut-free proof.



1
=
(0 ∨ 1) a (1 ∨ 0)
∨ă
(0 a 1) ∨ 1 a
0
ω₁
0 b 1
∧
(1 ∨ 0) a (0 ∨ 1)
∨ă
(1 a 0) ∨
0
ω₂
1 b 0
a
=
1
1 b 1
bă
=
1 a 1
1
b (0 a 1)
∨∧̌
cut
(0 a 1) ∧ (1 a 0)
(0 ∧ 1) a (1 ∧ 0)
∨ (1 a (0 b 1)) ∨ (1 b (0 a 1))
=
(1 a (0 b 1)) ∨ (1 b (0 a 1))
1
=
(0 ∨ 1) ∧ 1 ∨
0
l_a ω₂
1 b 0
∨∧̌
(0 ∧ 1) ∨ (1 ∨ (1 b 0))
=
1 ∨ (1 b 0)
1
=
1 ∨
0
r_a ω₁
0 b 1
∧ 0 ∨
=
1
1 b 1
∨∧̌
(1 ∧ 0) ∨ ((0 b 1) ∨ (1 b 1))
=
((0 b 1) ∨ (1 b 1))



- For every formula B, there exists a cut-free derivation: $\begin{array}{c} l_{\mathsf{a}} B \ \mathsf{a} \ r_{\mathsf{a}} B \\ \| \\ B \end{array}$

$$
= \cfrac{1}{\cfrac{\boxed{= \cfrac{1}{(0 \vee 1) \;\wedge\; \boxed{1 \vee \boxed{\begin{matrix} 0 \\ l_{a}\omega_{2} \,\| \\ 1\ \mathsf{b}\ 0 \end{matrix}}}} \;\; \vee\check{\wedge} \;\; = \cfrac{(0 \wedge 1) \vee (1 \vee (1\ \mathsf{b}\ 0))}{1 \vee (1\ \mathsf{b}\ 0)}} \;\; \mathsf{a} \;\; \boxed{= \cfrac{1}{\boxed{1 \vee \boxed{\begin{matrix} 0 \\ r_{a}\omega_{1} \,\| \\ 0\ \mathsf{b}\ 1 \end{matrix}}} \;\wedge\; \boxed{0 \vee \boxed{= \cfrac{1}{1\ \mathsf{b}\ 1}}}} \;\; \vee\check{\wedge} \;\; = \cfrac{(1 \wedge 0) \vee ((0\ \mathsf{b}\ 1) \vee (1\ \mathsf{b}\ 1))}{((0\ \mathsf{b}\ 1) \vee (1\ \mathsf{b}\ 1)}}}{\vee\check{a} \;\; (1\ \mathsf{a}\ (0\ \mathsf{b}\ 1)) \;\vee\; \boxed{\mathsf{b}\check{a}\ \cfrac{(1\ \mathsf{b}\ 0)\ \mathsf{a}\ (1\ \mathsf{b}\ 1)}{\boxed{= \cfrac{1\ \mathsf{a}\ 1}{1}}\ \mathsf{b}\ (0\ \mathsf{a}\ 1)}}}}
$$

Conclusion

- Given connectives α and β, we define dual instances of the shape: the *up* and *down* rules

$$\alpha\hat{\beta}\,\frac{(A\ \beta\ B)\ \alpha\ (C\ \hat{\beta}\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \alpha\ D)} \qquad \beta\check{\alpha}\,\frac{(A\ \beta\ B)\ \alpha\ (C\ \beta\ D)}{(A\ \alpha\ C)\ \beta\ (B\ \check{\alpha}\ D)}$$

- System DT^{sa} discovered via the subatomic methodology
- Defined as the natural 'completion' of system SKS^{sa} : generated by all rules rather than a subset of rules, for $\alpha, \beta \in \{\wedge, \vee\} \cup \mathcal{A}$
- Thus, system DT^{sa} really is *just* the shape!
- Adding more rules gets us a system even simpler than classical propositional logic
- Proof of cut elimination becomes a triviality

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- Benjamin Ralph, *Modular normalisation of classical proofs*, Ph.D. thesis, University of Bath, 2019.
- Andrea Aler Tubella and Alessio Guglielmi, *Subatomic proof systems: Splittable systems*, ACM Transactions on Computational Logic (TOCL) 19 (2018).
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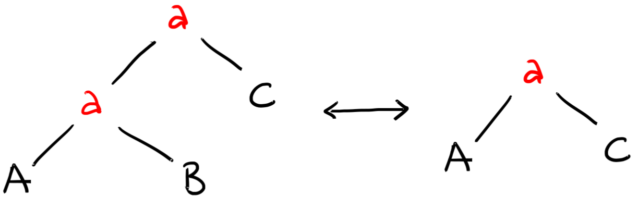
DT Weakenings

- It is possible to introduce redundancy into decision trees



- For every formula A, B, C and D and atom $\mathsf{a} \in \mathcal{A}$, we can construct cut-free derivations which we call *DT-weakenings*:

$$\begin{array}{c} A \;\mathsf{a}\; C \\ \| \\ (A \;\mathsf{a}\; B) \;\mathsf{a}\; C \end{array} \qquad \begin{array}{c} A \;\mathsf{a}\; (B \;\mathsf{a}\; C) \\ \| \\ A \;\mathsf{a}\; C \end{array}$$

Cut Elimination: Construction

Lemma

For every formula B, there exists a cut-free derivation:

$$\begin{array}{c} l_a B \text{ a } r_a B \\ \Vert \\ B \end{array}$$

Idea: Reading bottom to top, re-order B using invertible inferences so that atom a is at the root, eliminating any redundant copies of the atom a using DT-weakenings.

Proof.

Structural induction on B:

If we have that $B \equiv (C\ \beta\ D)$, for $\beta \neq a$ (thus $l_a B = l_a C\ \beta\ l_a D$, $r_a B = r_a C\ \beta\ r_a D$), construct:

$$\beta\breve{a}\ \frac{(l_a C\ \beta\ l_a D)\ a\ (r_a C\ \beta\ r_a D)}{\boxed{\begin{array}{c} l_a C\ a\ r_a C \\ \phi \Vert \\ C \end{array}}\ \beta\ \boxed{\begin{array}{c} l_a D\ a\ r_a D \\ \psi \Vert \\ D \end{array}}}$$

Cut Elimination: Construction

Lemma

For every formula B, there exists a cut-free derivation:

$$\begin{array}{c} l_a B \text{ a } r_a B \\ \| \\ B \end{array}$$

Proof.

In the remaining case that $B \equiv (C \text{ a } D)$ (thus $l_a B = l_a C$, $r_a B = r_a D$), we can construct:

$$\begin{array}{c} l_a C \ a \ r_a D \\ \omega \| \\ \boxed{\boxed{\begin{array}{c} l_a C \ a \ r_a C \\ \phi \| \\ C \end{array}} \ a \ \boxed{\begin{array}{c} l_a D \ a \ r_a D \\ \psi \| \\ D \end{array}}} \end{array}$$

where ω is two instances of DT-weakening. □